An Isochron-Based Solution to the Target Defense Game Against a Faster Invader

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***Abstract*—This letter studies a target defense game between two defenders and a faster moving invader. We propose a novel approach that solves the saddle point strategy from a family of time-to-capture isochrons. We show that the game can be viewed as a pursuit-evasion game between the invader and the isochrons, where the latter are controlled by the defenders. The optimal defending strategy is the one that forces the invader to isochrons with shorter time-to-capture most efficiently. The proposed method is validated in a special case with zero capture range, where we prove that the saddle point strategy is optimal regardless of other players' behaviors. In addition, the isochrons are shown to be semipermeable, and the barrier of the target defense game is solved.**

***Index Terms*—Target defense, differential game, isochrons, barrier, counter-UAV.**

I. Introduction

THE TARGET defense game is an extension of the pursuit-evasion game (PEG) where the invader plays as the evader, while the defender as the pursuer [1]. In addition, the invader not only needs to avoid capture, but also seeks to enter a target area [2], [3]. This game is an abstraction of the counter-UAV mission where a prohibited area must be protected against hostile drones [4].

In PEGs, a critical parameter is the player speed, as it affects the solution drastically. For example, a pursuer is guaranteed to capture at least one evader if the latter travels slower [5], [6]. For faster evaders, however, at least two pursuers are required to capture a single evader [7]. An extensively studied faster evader problem is for the evader to break out an encirclement of defenders [8]. To do this, the evader must approach the defenders actively, which allows an opportunity for capture. For the target defense game, the invader needs to sneak inside the encirclement instead of getting out [9].

To solve a PEG problem, the standard approach is to use the Hamilton-Jacobi-Bellman-Isaacs (HJI) equation [10]. For a target defense game, an extended HJI equation can be constructed by combining the two goals of the invader into a single objective function [11], [12].

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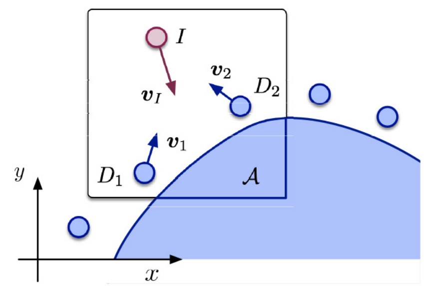

Fig. 1. The target defense game.

Solving the HJI equation is a challenging task, the computation cost increases significantly with system dimensions [13]. In some cases, the HJI equation can be simplified by breaking down the problem into subsystems [14]. In [15] the HJI equation of a faster evader game is decomposed and solved with a smart choice of reference frame. The result in [15] has been extended into a target defense game by adding a circular target [16]. However, the solution is obtained backward in time. In addition, due to the special structure of this solution, obtaining a state-feedback strategy that is easy to use in practice is not a trivial task. Szőts and Harmati suggest to extract the state-feedback strategy numerically, but extensive computation is required [17]. Fu and Liu propose a geometric defending strategy, but its optimality is not guaranteed [18].

Instead, this letter proposes to develop a computational-efficient forward-looking state-feedback strategy by reformulating the problem into a PEG between the invader and a family of isochrons. This isochron-PEG has a linear objective, therefore has an analytical solution.

The proposed method is verified in a special case with zero capture range, where the isochrons have a simple form. We prove the optimality of the resultant strategy, and solve the barrier of the game.

II. Problem Statement

A convex target area $\mathcal{A} = \{\boldsymbol{x} | g(\boldsymbol{x}) \leq 0\}$ is surrounded by multiple defenders, their goal is to prevent the invader I from entering the target. On the other hand, if the invader is to enter, it must pass two defenders in between. Let D_1, D_2 be the two defenders in concern. This situation is shown in Fig. 1.

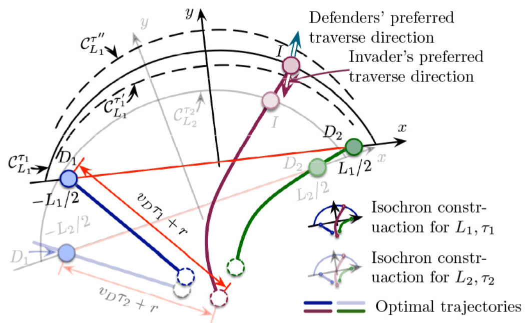

Fig. 2. Isochron-PEG formulation. Two groups with deep and shallow colors show two exemplar isochron constructions for different L, τ. For fixed L, e.g., $L = L_1$, different choices of τ admit different isochrons. For $\tau_1' > \tau_1 > \tau_1''$, we have $\mathcal{C}_{L_1}^{\tau_1''}$ above $\mathcal{C}_{L_1}^{\tau_1}$ above $\mathcal{C}_{L_1}^{\tau_1'}$.

Assume the game space is sufficiently large, i.e., distances traveled by the players are much larger than their physical sizes, so the velocity command can be followed within relatively short time. As commonly does in differential game literature, to simplify the HJI equation, we represent the vehicle movements as single integrators:

$$\begin{aligned} \dot{\boldsymbol{x}}_I &= \boldsymbol{v}_I, \ \ \|\boldsymbol{v}_I\| \le v_I, \\ \dot{\boldsymbol{x}}_j &= \boldsymbol{v}_j, \ \ \|\boldsymbol{v}_j\| \le v_D, \ \ j = 1, 2 \end{aligned} \tag{1}$$

where $\boldsymbol{x}_I$, $\boldsymbol{x}_j$ are locations of the invader and the jth defender, $\boldsymbol{v}_I$, $\boldsymbol{v}_j$ are the velocities and also the control inputs. v_I and v_D are the speed limits, $v_I > v_D$. The invader is captured when $\exists j \in \{1, 2\}$ such that $\|\boldsymbol{x}_I - \boldsymbol{x}_j\| < r$, where r is a predefined value called the capture range. The invader wins the game by entering the target without capture, the defenders win by capturing the invader before it enters.

Technically, optimal trajectories of this game have been solved in [16]. However, these trajectories are generated backward in time, i.e., can only be computed when terminal locations are known. To play the game in real time, one must predict how the game can end using instantaneous locations. This is a challenging task due to the special structure of the optimal trajectory. Instead, this letter proposes to re-formulate the problem into a PEG using isochrons, which gives rise to a state-feedback strategy that is easier to use in practice.

Without loss of generality, let D_2 be the defender that is closer to the invader, i.e., $\|ID_2\| \le \|ID_1\|$. As described in [16] and illustrated in Fig. 2, optimal trajectories of the invader (red) and D_2 (green) are closely related and must be solved together, while the trajectory of D_1 (blue) is a straight line with multiple possible orientations. These trajectories are determined by two parameters, denoted by τ and Γ, where τ has a clear physical meaning, the expected time to capture.

Define reference frame $\mathcal{F}_{D_1D_2}$, where the x-axis is along vector $\overrightarrow{D_1D_2}$, and the y-axis is along its bisector, pointing to the invader's side. Let $L = \|D_1D_2\|$, in $\mathcal{F}_{D_1D_2}$, an isochron can be constructed for a properly chosen (L, τ) pair.

We start by finding one point on the isochron. To do this, first pick an achievable terminal location for each player, and select a value for Γ. With τ and Γ, the optimal trajectories of I and D_2 can be solved, which give initial locations of I and D_2. Then the initial position of D_1 can be solved using two constraints, 1) its distance to D_2's initial location is L (red segments along x-axes in Fig. 2), and 2) its distance to I's terminal location is $v_D\tau + r$ (red double-arrow segments). With a reference frame conversion, the invader's location in $\mathcal{F}_{D_1D_2}$ can be obtained.

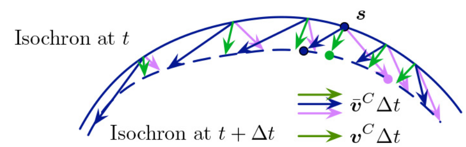

Fig. 3. Propagation speed of the isochron.

With L and τ fixed, we can change the value of Γ to obtain a set of invader locations. These locations form a curve, $\{(x, y)|f(x, y, L, \tau) = 0\}$, denoted by $\mathcal{C}_L^\tau$. By construction, invaders starting from $\mathcal{C}_L^\tau$ will be captured exactly after time period τ, hence we refer to $\mathcal{C}_L^\tau$ as an isochron. Apparently, the isochrons move with defenders.

During the game, the invader may traverse the isochrons. Roughly speaking, the defenders wish the traverse result in smaller τ, or equivalently, follow positive y in frame $\mathcal{F}_{D_1D_2}$, while the invader prefers the opposite. See Fig. 2.

This traverse preference can be viewed as the invader chasing after the isochrons, so we obtain an isochron-PEG with the invader as the pursuer, and the defenders as evaders.

III. The Isochron-PEG

Defender motions contribute to the isochron movement in three aspects, linear motion $\boldsymbol{v}^L$, rotation $\boldsymbol{v}^R$, and propagation $\boldsymbol{v}^C$. Viewing from frame $\mathcal{F}_{D_1D_2}$, we have

$$\boldsymbol{v}^L = \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2)$$

Let $\boldsymbol{v}_i = (v_{ix}, v_{iy})$, $i = 1, 2$, the rotational speed is

$$\boldsymbol{v}^R = \frac{1}{L}(v_{2y} - v_{1y})(-y, x)^T$$

To compute the propagation velocity, first take time derivative of the isochron function $f(x, y, L, \tau) = 0$ and obtain

$$f_x v_x + f_y v_y + f_L v_L + f_\tau \frac{d\tau}{dt} = 0 \tag{2}$$

where $v_L = v_{2x} - v_{1x}$ is the changing rate of L, and $d\tau/dt = -1$ because τ decreases at the same rate as the time progress.

Given defender velocities, v_L can be computed, but v_x and v_y cannot be determined using (2) only. Let $(v_x, v_y)^T$ that satisfies (2) be a candidate propagation velocity, denoted by $\bar{\boldsymbol{v}}^C$.

As shown in Fig. 3, suppose the isochron moves during time period Δt, and a point on it propagates from $\boldsymbol{s}$ to $\boldsymbol{s} + \bar{\boldsymbol{v}}^C \Delta t$. The latter can take multiple values, but we are interested in the one that admits the most efficient propagation, i.e., $\boldsymbol{v}^C = \arg\min_{\bar{\boldsymbol{v}}^C} \|\bar{\boldsymbol{v}}^C\|$. Solution to this problem is

$$\boldsymbol{v}^C = \boldsymbol{c} + \boldsymbol{B}\boldsymbol{v}_D \tag{3}$$

where

$$\begin{aligned} \boldsymbol{B} &= \begin{pmatrix} f_x & 0 & -f_x & 0 \\ f_y & 0 & -f_y & 0 \end{pmatrix} \frac{f_L}{f_x^2 + f_y^2} \\ \boldsymbol{c} &= \frac{f_\tau}{f_x^2 + f_y^2} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ \boldsymbol{v}_D &= \begin{pmatrix} \boldsymbol{v}_1^T & \boldsymbol{v}_2^T \end{pmatrix}^T \end{aligned}$$

Denote by $\boldsymbol{e}^C$ the unit vector along the gradient of f,

$$\boldsymbol{e}^C = \frac{1}{\sqrt{f_x^2 + f_y^2}} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \tag{4}$$

we find $\boldsymbol{v}^C$ is colinear with $\boldsymbol{e}^C$. More precisely,

$$\boldsymbol{v}^C = \text{sign}(f_L v_L - f_\tau)\|\boldsymbol{v}^C\|\boldsymbol{e}^C$$

For the defenders, $\boldsymbol{v}^C$ is the acceptable approaching speed of the invader. On the other hand, the actual invader velocity in frame $\mathcal{F}_{D_1D_2}$ is $\boldsymbol{v}_I' = \boldsymbol{v}_I - (\boldsymbol{v}^L + \boldsymbol{v}^R)$, or in matrix form,

$$\boldsymbol{v}_I' = \boldsymbol{v}_I + \boldsymbol{A}\boldsymbol{v}_D \tag{5}$$

where

$$\boldsymbol{A} = \begin{pmatrix} -1/2 & -y/L & -1/2 & y/L \\ 0 & -1/2 + x/L & 0 & -1/2 - x/L \end{pmatrix}$$

As mentioned in Section II, the competition of this game is on the invader's traverse direction across the isochrons. This direction can be represented by the difference between $\boldsymbol{v}_I'$ and $\boldsymbol{v}^C$, i.e., by $\Delta\boldsymbol{v} = \boldsymbol{v}_I' - \boldsymbol{v}^C$. However, points on the same isochron have the same time-to-capture, hence we are only concerned of the component of $\Delta\boldsymbol{v}$ along $\boldsymbol{e}^C$. Since $\boldsymbol{v}^C$ is colinear with $\boldsymbol{e}^C$, we can define the objective as

$$J = \Delta\boldsymbol{v}^T\boldsymbol{e}^C = (\boldsymbol{v}_I')^T\boldsymbol{e}^C - \|\boldsymbol{v}^C\|$$

Here we use the assumption that $f_L v_L - f_\tau > 0$. This can easily be achieved by defenders allocating enough velocities toward each other. A rigorous proof will be presented for the case study in Section IV.

By construction, defenders' (invader's) preferred traverse directions are reflected by $J < 0$ ($J > 0$), hence the isochron-PEG can be described by a minmax problem:

$$\min_{\boldsymbol{v}_1,\boldsymbol{v}_2} \max_{\boldsymbol{v}_I} J$$

To solve it, we need an explicit form of J. Using (3)-(5) and some manipulations, J becomes

$$J = \frac{\boldsymbol{p}_1^T}{\sqrt{f_x^2 + f_y^2}}\boldsymbol{v}_1 + \frac{\boldsymbol{p}_2^T}{\sqrt{f_x^2 + f_y^2}}\boldsymbol{v}_2 + \boldsymbol{v}_I^T\boldsymbol{e}^C + \frac{f_\tau}{\sqrt{f_x^2 + f_y^2}}$$

where

$$\boldsymbol{p}_1 = \left(\frac{f_x}{2} + f_L, \frac{f_y}{2} + \frac{yf_x - xf_y}{L}\right)^T$$

$$\boldsymbol{p}_2 = \left(\frac{f_x}{2} - f_L, \frac{f_y}{2} - \frac{yf_x - xf_y}{L}\right)^T$$

So J is a linear combination of four terms. Except the last one, each term is controlled by a single player. So we can define an independent objective for each player:

$$J_j = \frac{\boldsymbol{p}_j^T\boldsymbol{v}_j}{\sqrt{f_x^2 + f_y^2}}, j = 1, 2$$

$$J_I = \boldsymbol{v}_I^T\boldsymbol{e}^C$$

As a result, the overall objective becomes

$$J = J_1 + J_2 + J_I + \frac{f_\tau}{\sqrt{f_x^2 + f_y^2}}$$

By ignoring the constant term, the game becomes

$$\min_{\boldsymbol{v}_1} J_1 + \min_{\boldsymbol{v}_2} J_2 + \max_{\boldsymbol{v}_I} J_I \tag{6}$$

Equation (6) is composed of three independent linear optimization problems, each has an analytical solution. For defenders,

$$\boldsymbol{v}_j^* = -v_D\frac{\boldsymbol{p}_j}{\|\boldsymbol{p}_j\|}, \ j = 1, 2 \tag{7}$$

For the invader,

$$\boldsymbol{v}_I^* = v_I\boldsymbol{e}^C \tag{8}$$

Equations (7), (8) are also called saddle point strategies.

In (7), (8), $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{e}^C$ are all determined by partial derivatives of the isochron function $f(x, y, L, \tau)$, i.e., f_x, f_y and f_L. To compute them, we need to determine τ. This can be solved from $f(x, y, L, \tau) = 0$, where (x, y) is the invader's location in $\mathcal{F}_{D_1D_2}$, L is the distance between the two defenders. Both (x, y) and L can be computed using player locations. This connects $\boldsymbol{v}_j^*$ $\boldsymbol{v}_I^*$ directly to player locations, making (7), (8) state-feedback strategies.

Note strategy (7) is always optimal for defenders, regardless of the invader's strategy. The same claim holds for (8).

We conclude the discussion above into the theorem below.

Theorem 1: For a two-defender single-invader target defense game with kinematics (1) and $v_D < v_I$, assume a set of isochrons exist, each represented by $f(x, y, L, \tau) = 0$, where f is differentiable, (x, y) is the invader's location in $\mathcal{F}_{D_1D_2}$, L is the distance between the two defenders, τ is the expected time to capture. Then the optimal defending (invading) strategy is given by (7), (8) regardless of the invader's (defenders') behavior.

IV. Case Study: Point Capture Problem

The derivation in Section III makes no assumption on the capture range, therefore the proposed solution is suitable for any capture range, as long as the isochrons are solved.

In this section, the proposed approach is validated in a special case with zero capture range (point capture problem), because its isochrons are analytical [16]:

$$f(x, y, L, \tau) = x^2 + \left(y + \sqrt{v_D^2\tau^2 - \frac{L^2}{4}}\right)^2 - v_I^2\tau^2 \tag{9}$$

For a given L, τ takes value in $[\tau_{\min}^L, \tau_{\max}^L]$, where $\tau_{\min}^L = L/2v_D$, $\tau_{\max}^L = L/2\sin\gamma_0 v_D$, $\gamma_0 = \arccos(v_D/v_I)$ [16]. With this and the isochron function (9), we can further determine the limits for x and y:

$$|x| \leq x_{\max}^L = v_I\tau_{\min}^L \tag{10}$$

$$y_{\min}^{L,x} = \sqrt{\left(1 - \frac{v_D^2}{v_I^2}\right)\left(v_I^2(\tau_{\min}^L)^2 - x^2\right)} \tag{11a}$$

$$y_{\max}^{L,x} = \sqrt{v_I^2(\tau_{\min}^L)^2 - x^2} \tag{11b}$$

Equation (11a) is the barrier of the PEG counterpart of our target defense game, denoted by $\mathcal{B}$. The area beneath it, $\{(x, y)||x| \leq x_{\max}^L, 0 < y < y_{\min}^{L,x}\}$, is the invader's winning region, where no solution for τ exists, and the invader won't be captured starting from there. This solution agrees with [7] and [16]. Another form of the barrier is presented in (23b).

Suppose (x, y, L) is known from player locations, and τ is solved by letting (9) $= 0$. To solve the optimal strategy, first compute partial derivatives of f,

$$f_x = 2x \tag{12a}$$

$$f_y = 2\left(y + \sqrt{v_D^2\tau^2 - \frac{L^2}{4}}\right) \tag{12b}$$

$$f_L = -\frac{L}{2}\left(\frac{y}{\sqrt{v_D^2\tau^2 - L^2/4}} + 1\right) \tag{12c}$$

$$f_\tau = 2v_D^2\tau\left(\frac{y}{\sqrt{v_D^2\tau^2 - L^2/4}} + 1\right) - 2v_I^2\tau \tag{12d}$$

then calculate $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$,

$$\begin{aligned}\boldsymbol{p}_1 &= \left(\frac{f_x}{2} + f_L, \frac{f_y}{2} + \frac{yf_x - xf_y}{L}\right)^T = p_1\bar{\boldsymbol{p}}_1\\ \boldsymbol{p}_2 &= \left(\frac{f_x}{2} - f_L, \frac{f_y}{2} - \frac{yf_x - xf_y}{L}\right)^T = p_2\bar{\boldsymbol{p}}_2\end{aligned} \tag{13}$$

where

$$p_j = x + (-1)^j\frac{L}{2}\left(\frac{y}{\sqrt{v_D^2\tau^2 - L^2/4}} + 1\right)$$

$$\bar{\boldsymbol{p}}_j = \left(1, (-1)^j\frac{2\sqrt{v_D^2\tau^2 - L^2/4}}{L}\right), \ j = 1, 2$$

Because $\bar{\boldsymbol{p}}_1$ is in the fourth quadrant, $\bar{\boldsymbol{p}}_2$ is in the third quadrant, to determine the direction of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, we only need to examine p_1 and p_2. To do this, starting from the upper bound of x in (10), we have

$$x^2 \le (v_I\tau_{\min}^L)^2 + \left(\frac{L^2}{4}\frac{x^2}{v_D^2\tau^2} - \frac{L^2}{4}\frac{x^2}{v_D^2\tau^2}\right)$$

Substitute $\tau_{\min}^L = L/2v_D$ and re-organize, we have

$$|x|\sqrt{v_D^2\tau^2 - \frac{L^2}{4}} \le \frac{L}{2}\sqrt{v_I^2\tau^2 - x^2} \tag{14}$$

On the other hand, we can write y as a function of (x, L, τ) using (9) $= 0$, then p_j can be written as

$$p_j = x + (-1)^j\frac{L}{2}\frac{\sqrt{v_I^2\tau^2 - x^2}}{\sqrt{v_D^2\tau^2 - L^2/4}}, j = 1, 2$$

With this and (14) we can see that $p_1 \le 0$, $p_2 \ge 0$.

For the invader, it is not hard to compute

$$\boldsymbol{e}^C = -\left(x, y + \sqrt{v_D^2\tau^2 - \frac{L^2}{4}}\right)^T \tag{15}$$

With $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{e}^C$ given in (13) and (15), we can compute the optimal strategies using (7), (8). For defenders,

$$\boldsymbol{v}_j^* = \left((-1)^{j-1}\frac{L}{2\tau}, -\sqrt{v_D^2 - \frac{L^2}{4\tau^2}}\right), \ j = 1, 2 \tag{16}$$

For the invader,

$$\boldsymbol{v}_I^* = -\left(\frac{x}{\tau}, \frac{y + \sqrt{v_D^2\tau^2 - L^2/4}}{\tau}\right)^T. \tag{17}$$

Theorem 2: For the point capture problem, when saddle point strategies (16), (17) are applied, $J = J^* = 0$.

Proof: Under strategies (16), (17), the objective becomes

$$J = J^* = -v_D\frac{|\boldsymbol{p}_1| + |\boldsymbol{p}_2|}{\sqrt{f_x^2 + f_y^2}} + v_I + \frac{f_\tau}{\sqrt{f_x^2 + f_y^2}} \tag{18}$$

Note $|\bar{\boldsymbol{p}}_1| = |\bar{\boldsymbol{p}}_2| = 2v_D\tau/L$, we have

$$\begin{aligned}|\boldsymbol{p}_1| + |\boldsymbol{p}_2| &= \frac{2v_D\tau}{L}(|p_1| + |p_2|)\\ &= 2v_D\tau\left(\frac{y}{\sqrt{v_D^2\tau^2 - L^2/4}} + 1\right)\end{aligned} \tag{19}$$

With the isochron function (9) $= 0$ and partial derivatives (12a), (12b), we know

$$\sqrt{f_x^2 + f_y^2} = 2v_I\tau \tag{20}$$

Substitute (19), (20) and (12d) into (18), we have

$$\begin{aligned}J^* = &-v_D\frac{2v_D\tau}{2v_I\tau}\left(\frac{y}{\sqrt{v_D^2\tau^2 - L^2/4}} + 1\right) + v_I\\ &+ \frac{2v_D^2\tau}{2v_I\tau}\left(\frac{y}{\sqrt{v_D^2\tau^2 - L^2/4}} + 1\right) - \frac{2v_I^2\tau}{2v_I\tau} = 0.\end{aligned}$$

■

Theorem 2 shows that under saddle point strategies, the invader must stay on the same isochron. From the proof we can also conclude that, as long as the optimal defending (invading) strategy is adopted, defenders (invader) can ensure $J \le 0$ ($J \ge 0$), i.e., a player can at least guarantee the traverse is not along its opponent's preferred direction.

Curves with such property are said to be semipermeable [1]. Therefore we can claim that

Corollary 1: Isochrons (9) $= 0$ are semipermeable.

In the definition for J, we used assumption $f_Lv_L - f_\tau > 0$. Here we prove it to be true for the point capture problem.

Theorem 3: For the point capture problem, under the optimal defending strategy (16), $\sigma = f_Lv_L - f_\tau > 0$.

Proof: According to strategy (16), $v_L = -L/\tau$, therefore

$$\begin{aligned}\sigma &= -\left(2v_D^2\tau - \frac{L^2}{2\tau}\right)\left(\frac{y}{\sqrt{v_D^2\tau^2 - L^2/4}} + 1\right) + 2v_I^2\tau\\ &= -\frac{2}{\tau}\left[\sqrt{v_D^2\tau^2 - \frac{L^2}{4}}\left(y + \sqrt{v_D^2\tau^2 - \frac{L^2}{4}}\right) - v_I^2\tau^2\right]\end{aligned}$$

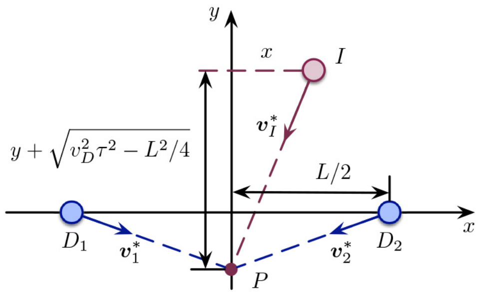

Fig. 4. Saddle point strategies for the point capture problem.

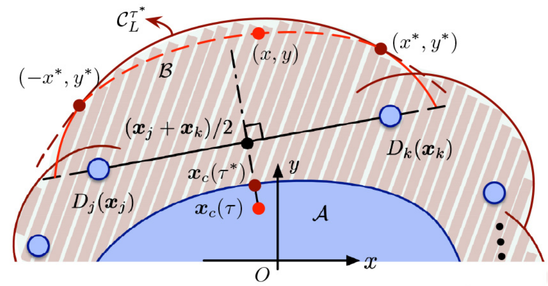

Fig. 5. Barrier of the point capture target defense game.

To prove $\sigma > 0$ is equivalent to proving

$$\sqrt{v_D^2\tau^2-\frac{L^2}{4}}\left(y+\sqrt{v_D^2\tau^2-\frac{L^2}{4}}\right) < v_I^2\tau^2 \tag{21}$$

A sufficient condition for (21) to hold is

$$\left(v_D^2\tau^2-\frac{L^2}{4}\right)+\left(y+\sqrt{v_D^2\tau^2-\frac{L^2}{4}}\right)^2 < 2v_I^2\tau^2$$

Use isochron function (9) $= 0$, we have

$$\left(v_D^2\tau^2-\frac{L^2}{4}\right) < 2x^2+\left(y+\sqrt{v_D^2\tau^2-\frac{L^2}{4}}\right)^2$$

which obviously holds. ■

Theorem 4: For the point capture problem, under saddle point strategies, an invader will be captured at a fixed point starting from the same isochron, and the optimal trajectories are straight lines.

Proof: Plot (x, y), $\boldsymbol{v}_1^*$, $\boldsymbol{v}_2^*$ and $\boldsymbol{v}_I^*$ in $\mathcal{F}_{D_1D_2}$, as shown in Fig. 4, it is obvious that $\boldsymbol{v}_1^*$, $\boldsymbol{v}_2^*$ and $\boldsymbol{v}_I^*$ all point to the same point, $P = (0, -\sqrt{v_D^2\tau^2 - L^2/4})$.

On the other hand, for $I \in \mathcal{C}_L^\tau$, we have

$$\frac{\|PI\|}{\|PD_i\|} = \frac{\sqrt{x^2+(y+\sqrt{v_D^2\tau^2-L^2/4})^2}}{v_D\tau} = \frac{v_I}{v_D} \tag{22}$$

Thus all the distances in Fig. 4 scale at the same rate as the game progresses. For example, after Δt, τ becomes $\tau' = \tau - \Delta t$. Let $k = \tau'/\tau$, we have $x' = kx$, $L' = kL$, $(\|PI\|)' = k\|PI\|$, etc. Then the new optimal velocities still point to P. As a result, the trajectories are straight lines, and the invader will eventually be captured at P. ■

Corollary 2: Given target $\mathcal{A} = \{\boldsymbol{x}|g(\boldsymbol{x}) \le 0\}$ and defender locations $\{\boldsymbol{x}_i\}_1^n$. For any adjacent defender pair (D_j, D_k), where $\overrightarrow{OD_k}$ is clockwise to $\overrightarrow{OD_j}$ and $g((\boldsymbol{x}_j+\boldsymbol{x}_k)/2) > 0$. The barrier expressed in frame $\mathcal{F}_{D_jD_k}$ is

$$\begin{cases} f(x, y, L, \tau^*) = 0, x \in [-x^*, x^*] & \text{(23a)} \\ x^2+\left(\dfrac{v_I y}{\sqrt{v_I^2-v_D^2}}\right)^2 = \left(\dfrac{v_I L}{2v_D}\right)^2, & \text{(23b)} \\ \quad x \in [-x_{\max}^L, -x^*,) \cup (x^*, x_{\max}^L] & \end{cases}$$

where $L = \|D_jD_k\|$, τ^* is the solution to

$$\begin{aligned} &\min_\tau \ |g(\boldsymbol{x}_c(\tau))| \\ &s.t.\ \ \tau_{\min}^L \le \tau \le \tau_{\max}^L \\ &\qquad \boldsymbol{x}_c(\tau) = \frac{1}{2}(\boldsymbol{x}_j+\boldsymbol{x}_k)+C_{-\pi/2}\frac{\boldsymbol{x}_k-\boldsymbol{x}_j}{\|\boldsymbol{x}_k-\boldsymbol{x}_j\|}\sqrt{v_D^2\tau^2-L^2/4} \end{aligned} \tag{24}$$

$C_{-\pi/2}(\boldsymbol{x}_k-\boldsymbol{x}_j)$ rotates $\boldsymbol{x}_k-\boldsymbol{x}_j$ by $\pi/2$ clockwise, $(\pm x^*, y^*)$ are the two intersection points of isochron $\mathcal{C}_L^{\tau^*}$ and the barrier of the PEG counterpart $\mathcal{B}$,

$$\begin{aligned} x^* &= \frac{v_I^2\tau}{v_D}\sqrt{\frac{(\tau_{\min}^L)^2}{\tau^2}+\frac{v_D^2}{v_I^2}-1} \\ y^* &= \left(\frac{v_I^2}{v_D^2}-1\right)\sqrt{v_D^2\tau^2-\frac{L^2}{4}} \end{aligned} \tag{25}$$

In addition, curve (23a) is smooth at $(\pm x^*, y^*)$.

Proof: Theorem 4 says that each isochron $\mathcal{C}_L^\tau$ is associated with a single capture location $\boldsymbol{x}_c(\tau)$, see Fig. 5.

For given target and defender locations, the neutral condition is $g(\boldsymbol{x}_c(\tau)) = 0$. Assume a solution for τ exists within $[\tau_{\min}^L, \tau_{\max}^L]$, denoted by τ^*. One can check that $(\pm x^*, y^*)$ are the two solutions to (9) $= 0$ and (11a) (or equivalently (23b)), and the two curves are tangent to each other there.

Consider point (x, y) on the barrier, and its time-to-capture τ. Use (9) one can check it holds that $\tau \ge \tau^*$ for $x \in [-x^*, x^*]$ and $\tau < \tau^*$ otherwise. According to Theorem 4, larger τ corresponds to smaller $g(\boldsymbol{x}_c(\tau))$. Because $g(\boldsymbol{x}_c(\tau^*)) = 0$, the segment of $\mathcal{B}$ within $[-x^*, x^*]$ should be replaced by $\mathcal{C}_L^{\tau^*}$.

If the defenders are far enough from the target such that $g(\boldsymbol{x}_c(\tau_{\max}^L)) > 0$, $\mathcal{B}$ is not affected. This situation is included in (24) and (25). ■

Corollary 2 shows the additional condition that the invader being captured outside of the target tends to enlarge its winning region. For multiple defenders spreading around the target, the joint invader winning region is the union of all, as shown by the shaded area in Fig. 5.

V. Simulation

To see the optimality of the proposed strategy, it is compared with its PEG counterpart [7], where strategies are represented by heading angles (see Fig. 6 for their definitions),

$$\phi_1^* = -\frac{\pi}{2}, \phi_2^* = \frac{\pi}{2} \tag{26a}$$

$$\cos\phi_I^* = \frac{d_1\sin\theta}{\sqrt{d_1^2+d_2^2-2d_1d_2\cos\theta}}$$

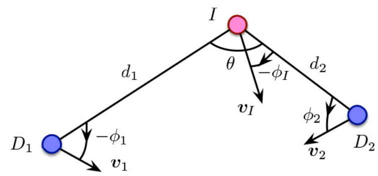

Fig. 6. Optimal strategies for the PEG counterpart.

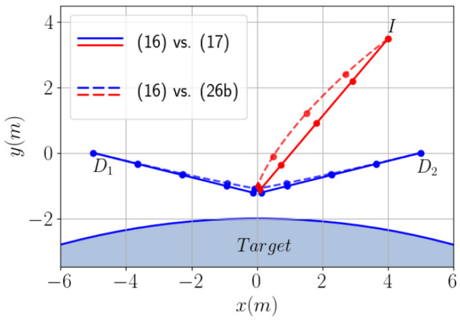

Fig. 7. Comparison between the PEG and target defense strategies: invader.

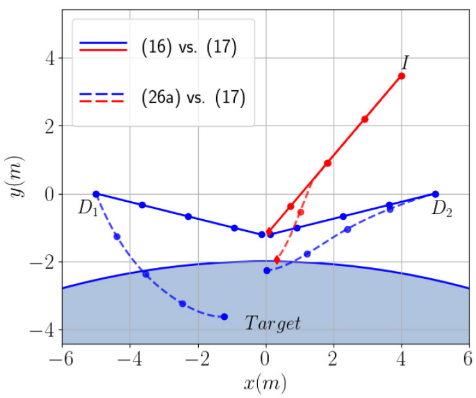

Fig. 8. Comparison between the PEG and target defense strategies: defender.

$$\sin \phi_I^* = \frac{d_1 \cos\theta - d_2}{\sqrt{d_1^2 + d_2^2 - 2d_1 d_2 \cos\theta}} \tag{26b}$$

where $\theta = \angle(\overrightarrow{ID_1}, \overrightarrow{ID_2})$, $d_j = \|\boldsymbol{x}_j - \boldsymbol{x}_I\|, j = 1, 2$.

In this strategy, (26a), (26b) ensures θ decreases (increases) in the most efficient way. The invader wins when $\theta > \pi$.

Fig. 7 compares (26b) and the saddle point invading strategy (17), under defenders' optimal play (16). Apparently, the invader is able to get closer to the target under (17).

Similarly, Fig. 8 compares (26a) and the optimal defending strategy (16), with the invader playing (17). It can be seen that (26a) successfully prevents the invader from passing in between the two defenders. However, this is achieved by the rotation of vector $\overrightarrow{D_1D_2}$. At the meantime, the defenders recede a large distance from their initial locations, so the invader is able to enter the target. This is an expected behavior of (26a), because it only focuses on θ. On the other hand, (16) doesn't ignore the competition on d_1, d_2, so the defenders are able to reduce these distances effectively and capture the invader before it enters the target.

This comparison means the proposed method provides a better strategy in terms of defending a target area.

VI. Conclusion

This letter proposes a novel concept to obtain an optimal state-feedback strategy for the target defense game with two defenders and a faster invader. We re-formulate the problem into a pursuit-evasion game between the invader and the time-to-capture isochrons. The proposed scheme is verified in the point capture problem. The resultant strategies are proved to be optimal and the winning region of the invader is solved.

The result of this letter is obtained under single integrator dynamics. It provides a reference for the best defense capability. For more complex systems, the proposed strategy can be implemented with a well-designed velocity controller.

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