Defending a Target Area With a Slower Defender

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***Abstract*—The target defense game is an abstraction of the counter-UAV mission, where a defender intends to intercept an invading drone before it enters a target area. While most studies on target defense games assume the defender travels faster, defending a target area with a slower defender is a less studied yet challenging problem, because capture cannot be guaranteed. This letter identifies two special cases where the defender has a chance to win, where the game region is bounded and where the target area is small. In the former case, the defender traps the invader at the corner. In the latter case, the defender delays the entering permanently by rotating around the target area at a sufficiently large angular speed. In both games, the optimal trajectory has a two-stage structure. Exploiting this feature, a novel method is proposed to solve for the barrier, which gives guidelines on how to deploy the defenders to ensure the target area been protected.**

***Index Terms*—Target defense, differential game, isochrons, barrier, counter-UAV.**

I. Introduction

THE RAPID development of unmanned aerial vehicles (UAVs) has not only created enormous opportunities, but also brought about increasing security risks. For example, a DJI quadrotor crashed near the White House in 2015 [1], a drone incident in 2018 at an airport of the United Kingdom caused a projected loss of more than 50 million Euro [2]. It has been reported that the number drone incidents is increasing every year [3]. The counter-UAV system is a technology that detects/intercepts hostile drones. Due to the increasing security concerns about drones, the counter-UAV market has been through a rapid expansion, at an estimated compound annual growth rate of 14.7% [4]. In this letter, we model the counter-UAV mission as a target defense differential game [5], where an invader seeks to enter the target area without being captured, while a defender intends to delay the entering or capture the invader before it enters.

The target defense game has been extensively studied. An analytical solution to defend a linear target in a rectangular region was proposed in [6]. An extension of the solution in an unbounded region was studied in [7]. A line defense game against cooperative invaders was solved in [8]. A variation with a moving line target was studied in [9]. A problem of maximizing the probability of detecting invaders crossing the boundary of the target area was solved in [10]. A set of perimeter defense games were studied in [11], [12], [13] where the defenders are restricted on the boundary of the target area. Reference [14] re-formulated the game as a path defending problem. Reference [15] integrated the target defense game into a path planning algorithm. A target defense game in the three dimensional space was studied in [16]. Target defense games are also addressed as reach-avoid games or reachability problems [17] and can be solved using numerical methods such as level-set method [18] and reinforcement learning [19].

While most of the studies assume the defender travels faster, the game is fundamentally different when it does not, because in such cases capture cannot be guaranteed [20], [21]. Target defense games with slower defenders are rarely studied. The maximum radius of a circular target that could be defended was solved in [22]. The optimal trajectory of this game was solved in [23]. A perimeter defense game was solved in [24]. A geometric strategy was designed in [25] for two cooperating defenders. An active target defense game with a faster invader was solved in [26].

A common strategy to handle a faster invader is to encircle it [27], [28], but this requires a large number of defenders. In this letter, we explore the possibility of defending a target area with a single slower defender. We identify two scenarios, where the game region is bounded (Game I), and where the target area is small (Game II). The main contribution of this letter is to solve the barriers, which reveal the winning conditions.

II. Problem Statement

A. Two Slower-Defender Games

1) Game I: Consider straight-line boundaries $x = 0$, $x = L_b$, and a linear target area $y \leq 0$, as shown in Fig. 1. The invader loses the game when hitting the boundaries, therefore the boundaries are also called the *deadlines*. In this game, the goal of the invader is to reach $y = 0$ and avoid the defenders and the deadlines, the goal of the defender is to capture the invader (at $y \geq 0$) with the help of the deadlines.

2) Game II: For simplicity, consider a circular target area. Here we emphasize that the size of the target area should be small, instead of the specific shape. More explanations can be found in [23]. Denote by R the radius of the target area, and let the center to be at $[0, 0]^T$, as shown in Fig. 2. Since there is no deadline to assist capture, the defender's goal in this game is to delay the invader's entering permanently. The goal of the invader is to enter the target area.

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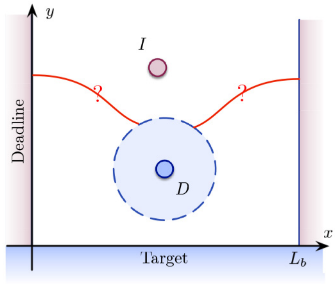


Fig. 1. Game I.

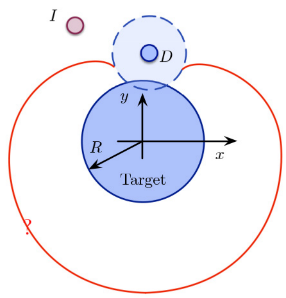


Fig. 2. Game II.

3) Dynamics, Goals, and Assumptions: For both games, we are interested in the defender's winning condition. i.e., given a defender location, does there exist a curve, called the *barrier* (red curves in Fig. 1 and Fig. 2), such that the invader cannot enter the target area when starts beyond. Both games follow the dynamic equation below:

$$\dot{x}_D = U_D \cos\phi, \quad \dot{y}_D = U_D \sin\phi,$$
$$\dot{x}_I = U_I \cos\psi, \quad \dot{y}_I = U_I \sin\psi_I, \quad (1)$$

where $U_D < U_I$ are the (constant) velocities of the players, ϕ, ψ are the heading angle (and also the control input) of the defender and the invader respectively. Let $a = U_D/U_I < 1$ be the speed ratio. *Capture* is defined as the distance between the two players being smaller than a positive *capture range*, i.e., $\|DI\| < r$, where $r > 0$.

B. Preliminaries

1) Barrier of a Game: For a two-player differential game, which player can win the game can be determined from the initial state if the *barrier* [5] is solved. For a game with n state variables, the *barrier* is an $(n-1)$-dimensional surface that divides the n-dimensional state space into two parts, each is the winning region of one player. If the game starts from the winning region of a certain player, this player is able to win the game regardless of the other player's behavior, as long as it applies its optimal strategy [29]. The barrier is composed of *optimal trajectories* that pass through terminal states with *neutral outcome*, so the goal of solving a differential game is to solve the barrier and the associated optimal strategies. Here *optimal trajectories* refer to the solution of (1) under optimal strategies, *neutral outcome* refers to the situation that no player wins the game.

2) Optimal Trajectories: The optimal trajectories of a slower-defender game has a two-stage structure. Assume the game starts with $\|DI\| > r$, the players will move along straight lines until $\|DI\|$ reduces to r. At this moment, if capture can be enforced, then the game ends. Otherwise the invader will start to maintain $\|DI\| = r$ by sparing part of its velocity to match that of the defender, and using the rest to rotate around, as shown in Fig. 3. This strategy is called the *distance maintaining strategy* [25]. The second stage is called the *loop-around stage*. It is at the invader's disposal whether to rotate around the defender. If the invader finds it possible to enter the target following a straight line, it will do so. Otherwise the loop-around stage lasts until a neutral terminal state is reached. If the latter is true, the corresponding optimal trajectory belongs to the barrier. Trajectories of the two stages connect smoothly.

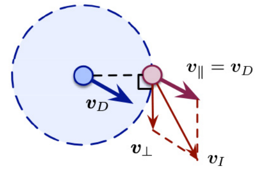


Fig. 3. Invader's distance maintaining strategy.

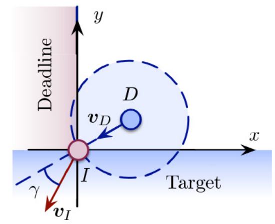


Fig. 4. Terminal states with neutral outcome of Game I.

Remark 1: When matching the defender's velocity in the loop-around stage, the invader needs access to the defender's heading angle. This can be measured from the defender's velocity. The same comment applies to Equation (4), which is the distance maintaining strategy applied to Game I.

III. Solution to Game I

A. Optimal Trajectories of Game I

1) Terminal States With Neutral Outcome: In Game I, terminal states with neutral outcome mean the invader is a) about to enter the target area while b) about to be captured. Requirement a) needs the invader to be captured at the boundary of the target area, requirement b) requires b1) capture also happens at the deadline and b2) the defender's velocity is along $\overrightarrow{DI}$. This is because if b1) is violated the invader can move horizontally to avoid capture, if b2) is violated the defender would have better chance of capture by allocating more velocity along $\overrightarrow{DI}$. a) and b1) implies capture to happen at the intersection of the target boundary and the deadline.

2) Existence of the Loop-Around Stage: Let γ be the invader's heading angle counted from $\overrightarrow{DI}$ as shown in Fig. 4. When the defender's velocity is along $\overrightarrow{DI}$, the relative velocity between the players along $\overrightarrow{DI}$ is $\Delta U = U_I \cos\gamma - U_D$, and the sign of ΔU is determined by γ. Let $\gamma^* = \arccos a$, the neutral terminal state can be reached with three possibilities, $\gamma > \gamma^*$ ($\Delta U < 0$), $\gamma = \gamma^*$ ($\Delta U = 0$), or $\gamma < \gamma^*$ ($\Delta U > 0$), among which only the second can be reached from the loop-around stage. In such case, $\|DI\| = r$ has been maintained for a period

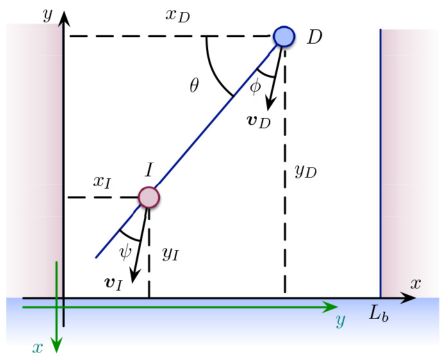


Fig. 5. Loop-around stage formulation of Game I.

of time. If $\gamma > \gamma^*$, the invader is captured immediately so the game does not have a loop-around stage.

3) Solution of the Loop-Around Stage: We simplify the loop-around stage by ignoring the deadlines and restoring them later. This simplification will be justified in Remark 3. Without the deadlines, the loop-around stage can be fully described by the invader's vertical coordinate y_I and the defender's relative bearing angle θ, as shown in Fig. 5. Then, the system dynamics becomes

$$\dot{\theta} = \frac{U_I}{r}\sin\psi - \frac{U_D}{r}\sin\phi, \tag{2a}$$

$$\dot{y}_I^{\text{loop}} = -U_I\sin(\theta+\psi), \tag{2b}$$

and the distance maintaining strategy requires

$$U_D\cos\phi = U_I\cos\psi. \tag{3}$$

We observe if the x, y axes are along the green directions in Fig. 5, dynamics (2) become identical to Isaacs' deadline game [5]. The neutral terminal state of the deadline game is $\gamma = \gamma^*$, hence the loop-around stage of Game I has the same solution as the deadline game except for 90° rotation. Since reference [5] only solved the problem using (θ, y_I), the solution in the inertial frame is presented below. According to [5], the optimal strategies are given by

$$\cos\phi = \frac{\cos\theta}{\sqrt{1+a^2-2a\sin\theta}}, \quad \sin\phi = \frac{a-\sin\theta}{\sqrt{1+a^2-2a\sin\theta}},$$
$$\cos\psi = \frac{a\cos\theta}{\sqrt{1+a^2-2a\sin\theta}}, \quad \sin\psi = \frac{1-a\sin\theta}{\sqrt{1+a^2-2a\sin\theta}}, \tag{4}$$

under which the bearing angle θ satisfies

$$\dot{\theta} = \frac{U_I}{r}\frac{1-a^2}{\sqrt{1+a^2-2a\sin\theta}}, \tag{5}$$

and the invader's position in the inertial frame satisfies

$$\dot{x}_I^{\text{loop}} = U_I\sin(\theta+\psi), \quad \dot{y}_I^{\text{loop}} = -U_I\cos(\theta+\psi). \tag{6}$$

Dividing (6) by (5) and substituting ψ from (4) gives

$$\frac{dx_I^{\text{loop}}}{d\theta} = \bar{r}\cos\theta, \quad \frac{dy_I^{\text{loop}}}{d\theta} = \bar{r}(\sin\theta - a), \tag{7}$$

where $\bar{r} = r/(1-a^2)$. If we place the original point at the invader's capture location, the integration of (7) gives

$$x_I^{\text{loop}} = \bar{r}(\sin\theta - \sin\gamma^*),$$
$$y_I^{\text{loop}} = -\bar{r}(\cos\theta - \cos\gamma^*) - a\bar{r}(\theta - \gamma^*). \tag{8a}$$

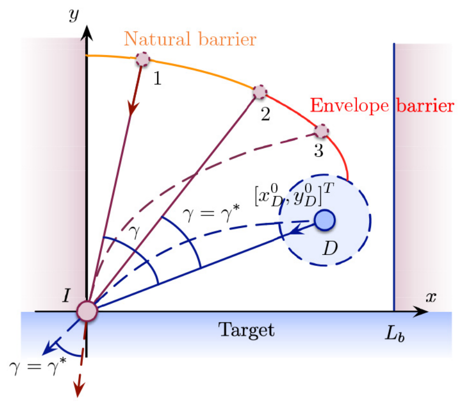


Fig. 6. Two parts of the barrier in Game I.

With the geometric relationship shown in Fig. 5, the defender's location can be computed as

$$x_D^{\text{loop}} = x_I^{\text{loop}} - r\sin\theta = \bar{r}(a^2\sin\theta - \sin\gamma^*),$$
$$y_D^{\text{loop}} = y_I^{\text{loop}} + r\cos\theta = -\bar{r}(a^2\cos\theta - \cos\gamma^*) - a\bar{r}(\theta - \gamma^*). \tag{8b}$$

4) Solution of the Optimal Trajectory: Equation (8) is parameterized by θ, which represents the switch point of the game's two stages. Derivatives of x_D^{loop}, y_D^{loop}, x_I^{loop}, y_I^{loop} w.r.t. θ give the tangent direction of the loop-around trajectory, which is also the direction of the straight-line trajectory since they connect smoothly. As a result, an analytical expression of the whole optimal trajectory can be obtained by taking derivatives of Equation (8) w.r.t. θ, multiplying the result by τ and adding them to Equation (8). Here τ is the second parameter of the optimal trajectory representing the time spent on the straight-line stage. The result is presented in Equation (9).

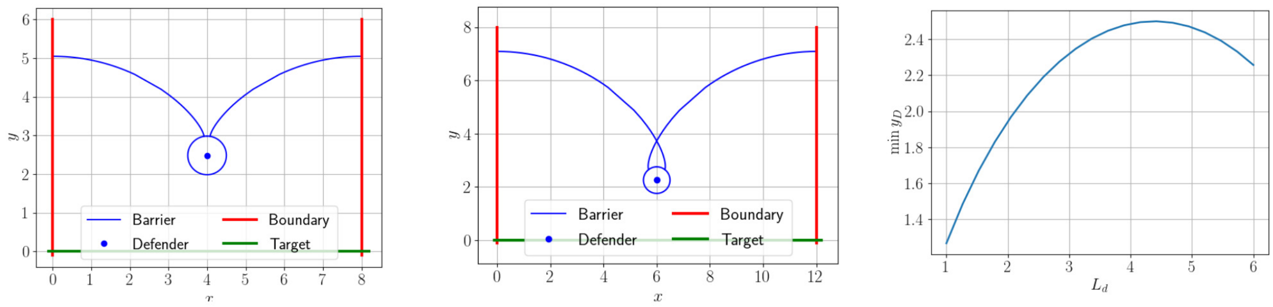
$$x_D(\theta,\tau) = x_D^{\text{loop}}(\theta) + a^2\tau\bar{r}\cos\theta,$$
$$y_D(\theta,\tau) = y_D^{\text{loop}}(\theta) + a\tau\bar{r}(a\sin\theta - 1), \tag{9a}$$
$$x_I(\theta,\tau) = x_I^{\text{loop}}(\theta) + \tau\bar{r}\cos\theta,$$
$$y_I(\theta,\tau) = y_I^{\text{loop}}(\theta) + \tau\bar{r}(\sin\theta - a). \tag{9b}$$

Remark 2: In the reference frame under which (9) is presented, the invader's final velocity is along the deadline.

B. Barrier of Game I

1) Natural Barrier: The fact that a loop-around stage may or may not exist divides the barrier of Game I into two parts. When no loop-around stage exists, the terminal state is reached from straight lines, therefore the barrier is an arc with radius $(\sqrt{(x_D^0)^2 + (y_D^0)^2} - r)/a$ centered at the capture location, given defender's initial location $[x_D^0, y_D^0]^T$. This part is called the *natural barrier*, as shown in Fig. 6. If the invader starts from the natural barrier, it will be captured with $\gamma > \gamma^*$. Initial locations of the invader from which it will be captured with $\gamma = \gamma^*$ are given by the *envelope barrier*.

2) Envelope Barrier: The envelope barrier must be solved from the optimal trajectory that contains a loop-around stage, which is Equation (9) for Game I. To do this, a novel method is proposed, the pseudo code is presented in Algorithm 1. This method is also applicable for Game II, and potentially to other slower-defender games with a loop-around stage.



(a) $L_d = 8$ (b) $L_d = 12$ (c) Relationship between y_D^0 and L_d

Fig. 7. Barrier of Game I. Parameters: $U_D = 1$, $U_I = 1.2$, $r = 0.5$.

Algorithm 1: Solve the Barrier

Data: The optimal trajectory, defender's initial location
Result: The envelope barrier

for *Allowable neutral terminal state* **do**
  Solve $(\theta, \tau) = (\theta^s, \tau^s)$ s.t. the defender's optimal trajectory passes the defender's initial location
  Compute the invader's location using (θ^s, τ^s), which gives one point on the envelope barrier
end

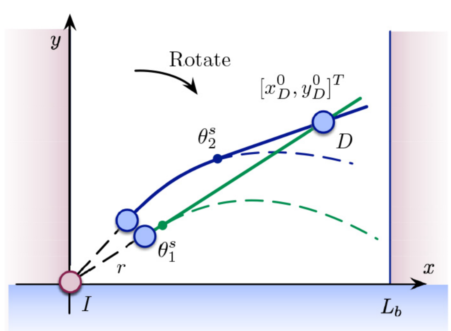


Fig. 8. A single solution of (θ, τ) exists for each rotation of the optimal trajectory. Green and blue are two exemplar rotations, whose solutions for θ are θ_1^s, θ_2^s.

For Game I, the allowable neutral terminal states refer to those satisfy the two requirements of Section III-A1. Note these requirements do not include the invader's final velocity, therefore different neutral terminal state can be obtained by rotating the optimal trajectory (9), as shown in Fig. 8. For each chosen terminal state, there exists a single solution of (θ, τ) for the defender's optimal trajectory to pass through the given $[x_D^0, y_D^0]^T$, denoted by (θ^s, τ^s). Shown in Fig. 8 are two examples of the rotation and the corresponding solutions for (θ^s, τ^s). After (θ^s, τ^s) is solved, the corresponding invader location can be computed using (9b), which gives one point on the envelope barrier. The complete envelope barrier can be solved through all allowable neutral terminal states. For Game I, the minimum rotation corresponds to $\tau^s = 0$, the maximum rotation corresponds to $\theta^s = \gamma^*$. The latter is also the end point of the natural barrier. Fig. 6 shows three exemplar points of the barrier, belonging to the natural barrier (point 1), the envelope barrier (point 3), and their intersection (point 2) respectively.

3) Barrier of Game I: Combining the natural and envelope barriers, an example of Game I's barrier is presented in Fig. 7. The (non-dimensional) parameters used are $r = 0.5$, $U_I = 1.2$, $U_D = 1$. It can be seen that the barrier and (a portion of) the defender's capture region divide the game region into two parts. The invader loses the game if starts from the upper part. Note the barrier has two branches due to symmetricity, which may intersect with each other, e.g., in Fig. 7(b). When intersect, only the parts above the intersection point belong to the actual barrier, and the parts below should be discarded.

The distance between the deadlines (L_b) and the distance between the defender and the target area (y_D^0) are related. For instance, in Fig. 7(a), $L_b = 8$, $y_D = 2.5$. In Fig. 7(b), $L_b = 12$, $y_D = 2.25$. Imagine the target area is moved downward while y_D^0 remains unchanged, the invader can be captured outside of the target area when starting from the barrier and above. However, if the target area is moved upward, the invader can enter the target area even starting from the barrier. This means

Proposition 1: In Game I, for fixed L_b, y_D^0 gives defender's minimum distance from the target area to win the game. Similarly, for given y_D^0, L_b is the maximum length of the target that can be defended. The relationship between y_D^0 and L_b is illustrated in Fig. 7(c).

Remark 3: Discussions above reveal how a slower defender can win Game I, to utilize the fact that the deadline cannot be crossed. Because the invader can only be captured at the deadline, the defender must push the invader toward it to the furthest extend. However, this requires the defender to recede towards the target area, and the best trade-off is given by the optimal strategy. Under this best trade-off, if the "furthest extent" reaches the deadline before the "receding" reaches the target, then the defender can win, so we can 1) solve for this trade-off and 2) check whether the deadline is at the desirable place. It can be seen that deadline is not involved in step 1). This justifies the simplification made in Section III-A3.

C. Barrier of the Deadline Game

We validate the proposed barrier solution method using Isaacs' deadline game [5]. In such game, the neutral terminal state requires the invader to be captured at any point along the deadline, and its velocity to be parallel to the deadline [5]. As a result, different allowable neutral terminal states correspond to the translation of the optimal trajectory instead of rotation, as shown in Fig. 9. Following Algorithm 1, the barrier of the deadline game is shown in Fig. 10. The solution

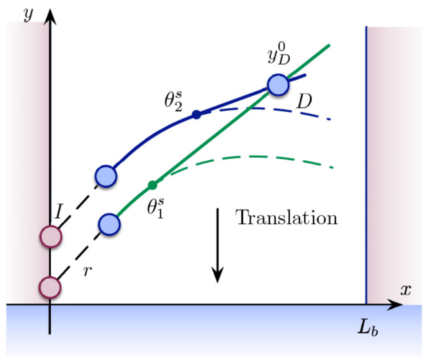


Fig. 9. Translation of the loop-around trajectory in the deadline game, each translation has a unique solution for (θ^S, τ^S).

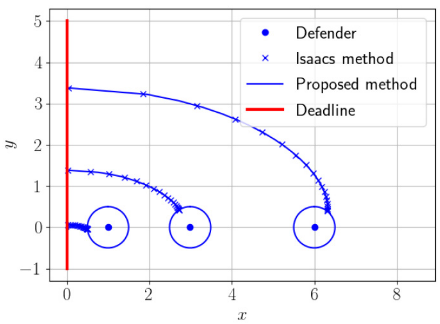


Fig. 10. Deadline game's barrier. Parameters: $U_D = 1$, $U_I = 1.2$, $r = 0.5$.

provided by [5] is also presented for comparison. The fact that the two solutions overlap justifies the proposed method. The (non-dimensional) parameters used in this game are $U_D = 1$, $U_I = 1.2$, $r = 0.5$.

IV. Barrier of Game II

A. Optimal Loop-Around Trajectories

1) When $R = 0$: This section applies Algorithm 1 on Game II. To do this, the loop-around optimal trajectory of Game II is needed, and it can be found in [23]. However, due to the specific method used, the solution does not distinguish different radii R of the target area, thus the game is equivalent to having a point target located at $[0, 0]^T$. In addition, the solution is in the polar coordinate, and the loop-around stage can be fully described by the distances of the two players from $[0, 0]^T$, denoted by ρ_D, ρ_I. The optimal loop-around trajectories of Game II are reproduced in Fig. 11. The dashed black lines are the constraint that the game stays in the loop-around stage.

In Fig. 11, we are interested in the region bounded by the solid blue, red and green curves. The solid blue line is the barrier of the loop-around subgame. If the game starts below this barrier, the invader can enter the target area by breaking the loop-around status. The red solid curve is the trajectory where the two players will end up rotating around the target area perpetually, therefore it is a stable equilibrium point of the system. In addition, this point is the only terminal state that entering can be prevented. At this point, $\gamma = \gamma^*$. Detailed explanations on Fig. 11 can be found in [23]. If the invader starts from between the red and blue lines, it will first reach

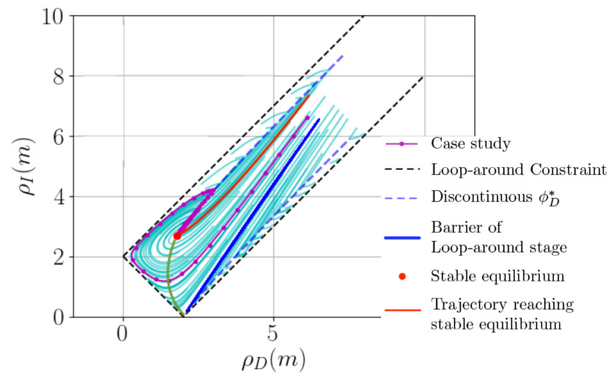


Fig. 11. Optimal loop-around trajectories of Game II in (ρ_D, ρ_I). Parameters: $r = 2$, $U_D = 1$, $U_I = 1.5$ [23].

a closest distance to the target area $\rho_I = \underline{\rho}_I$ before runs into the equilibrium point. The places where ρ_I attains its local minimum along a trajectory are represented by the green curve. Points on the green curve will be represented by $[\underline{\rho}_D, \underline{\rho}_I]^T$.

2) Distinguish Different R: Consider the trajectory that passes through $[\underline{\rho}_D, \underline{\rho}_I]^T$. If $R > \underline{\rho}_I$, the invader can enter the target. Therefore, such trajectory must be on the barrier of a specific game whose target area radius is $R = \underline{\rho}_I$.

B. The Barrier

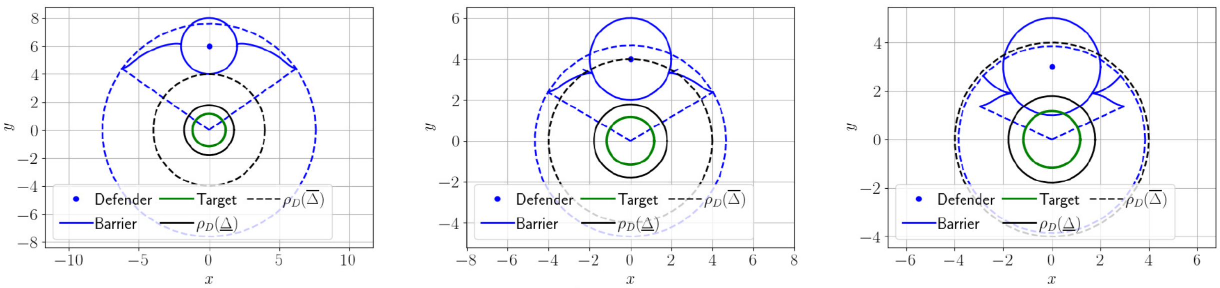
For a specific Game II with target area radius R, to apply Algorithm 1, the optimal trajectory required is the one with $\underline{\rho}_I = R$, the initial location of the defender is specified by ρ_D. Due to the symmetricity of the circular target area, the allowable neutral terminal states correspond to the rotation of the optimal trajectory w.r.t. the original point $[0, 0]^T$. An exemplar solution is shown in Fig. 12. Parameters of the game include $r = 2$, $U_D = 1$, $U_I = 1.5$, $R = 1.17$. The optimal trajectory used is shown in purple in Fig. 11. In Fig. 12, the barrier has extra branches as ρ_D decreases, the upper branches should be discarded.

Strictly speaking, the term "barrier" is not strict for the solid curves in Fig. 12, because they cease to move forward after reaching an end point and do not divide the space into separated regions. However, the barrier can be closed by adding extra defenders evenly around the target area, and the number of defenders needed depends on the central angle covered by a single defender. For example, the central angle covered in Fig. 12(a) is 107°, therefore 3 extra defenders are needed. For Game II, we have

Proposition 2: In Game II, for fixed ρ_D, the largest portion of the target area that can be defended is given by the central angle between the barrier's end points. Similarly, to defend a desired portion of the target area, a maximum ρ_D is needed. The relationship is presented in Fig. 13.

V. Conclusion

This letter studies two possible situations where a target area can be protected by a slower defender. The first game is bounded by two deadlines at the end points of a linear target area, the second game has a small enough circular target area. In both games, the optimal defending strategy is a balance between receding toward the target area and exploiting the



(a) $\rho_D = 6$ (b) $\rho_D = 4$ (c) $\rho_D = 3$

Fig. 12. Barrier of Game II. Parameters: $r = 2$, $U_D = 1$, $U_I = 1.5$, $R = 1.17$.

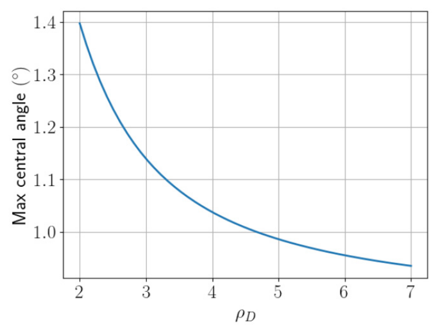


Fig. 13. ρ_D vs. the central angle defended by a single defender in Game II. Parameters: $r = 2$, $U_D = 1$, $U_I = 1.5$, $R = 1.17$.

topology of the environment. The barriers of the games reveal the relationship between the defender location and the range of the target area that can be protected, which can be used as guidelines to deploy the defenders.

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